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Interpolation and Approximation on the Unit Disc by Complex Harmonic Splines

HAN-LIN CHEN

Institute of Mathematics, Academia Sinica, Beijing, People's Republic of China

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1. INTRODUCTION

Complex analysis has its applications in many mathematical and physical problems, but it is, in most cases, very difficult to find out the exact representation of the functions (or solutions) to be considered in practice. This makes it important to search for effective ways of approximating the functions (or solutions) for which only finite data are given.

The Lagrange interpolating polynomial is unsuited for this task since one cannot ensure its convergence in all cases (see [3]). At times, a modification of the complex Lagrange interpolating polynomial may be used successfully (see [5]), but there is no general formula for this approach.

Ahlberg [1] uses analytic splines to approximate an analytic function on the unit disc, but these approximants only converge uniformly on compact subsets of the open disc.

Here we use the complex harmonic splines. They are obtained by using the Poisson integral, with the boundary function an interpolating or pseudo-interpolating complex spline. The complex harmonic splines converge uniformly on the closed unit disc to the analytic function being approximated as the mesh size tends to zero.

One would hope that the approximating functions has many of the properties of the function being approximated, e.g.,

(a) they are open mappings (conservation of domain);

(b) they are one-to-one mappings.

We prove that these properties do indeed hold for the complex harmonic spline in case the mesh size is sufficiently small.

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We illustrate by an example (see Fig. 1 at the end of this paper) that the complex harmonic spline approximates conformal maps in a satisfactory manner. Besides, a complex harmonic spline only consists of elementary functions.

2. Definitions and Notations

We denote by

$$U := \{Z: |Z| < 1\}$$

the (open) unit disc, and by Γ its boundary.

We denote by $\mathscr{S}_n(\Delta)$ the family of complex splines of degree n with (ordered) knot sequence $\Delta = (Z_1, ..., Z_M)$. Here, each Z_j is a point in Γ. We use $\Gamma_j := \widehat{Z_j Z_{j+1}}$ for the arc between the two points, and set $Z_{M+1} := Z_1$. We denote by $\mathscr{L}(F)$ the quasi-interpolant operator into complex splines introduced in [2]. We use $AC^{(n)}(\Gamma)$ for the class of functions in $C^{(n)}(\Gamma)$ with an absolutely continuous nth derivative.

Let D be a simply connected domain, with γ its boundary. A real function $u \in C^{(2)}(D)$ is said to be harmonic in D if it satisfies Laplace's equation $\Delta u = 0$ there. We call a function a complex harmonic function on D if it is a finite *complex* linear combination of harmonic functions in D, and denote the totality of all such functions by $H(D)$. These complex harmonic functions share with the (real) harmonic functions many properties, such as the mean-value theorem, the maximum modulus principle, Poisson formula, the Schwarz theorem, etc. We define

$$H^{(n)}(\bar{D}) := C^{(n)}(\bar{D}) \cap H(D)$$

and

$$AH^{(n)}(\bar{D}) := AC^{(n)}(\Gamma) \cap H^{(n)}(\bar{D}).$$

3. Results

The complex harmonic function

$$P(Z) = \frac{1}{2\pi} \int_0^{2\pi} p(\zeta) \operatorname{Re}\left(\frac{\zeta + Z}{\zeta - Z}\right) d\theta \tag{1}$$

is said to be a complex harmonic spline if p belongs to $\mathscr{S}_n(\Delta)$. The family of such functions is denoted by $SH(U)$.

Since $p(\zeta)=p_j(\zeta)\in\pi_n$, $\zeta\in\Gamma_j=\widehat{Z_jZ}_{j+1}$, $j=\overline{1,M}$, the expansion around the point Z in the ζ-plane is as follows:

$$p_j(\zeta)=a_n^{(j)}(\zeta-Z)^n+a_{n-1}^{(j)}(\zeta-Z)^{n-1}+\cdots+a_0^{(j)},\qquad \zeta\in\Gamma_j. \tag{2}$$

$a_n^{(j)}$ is obviously independent of Z.

Let $\sigma_k^{(j-1)}=a_k^{(j)}-a_k^{(j-1)}$, $k=0,\dots,n$, $\sigma_k^{(0)}=\sigma_k^{(n)}=a_k^{(1)}-a_k^{(n)}$, $k=0,\dots,n$. By induction we have

$$\sigma_{n-k}^{(j-1)}=(-1)^k\binom{n}{k}\sigma_n^{(j-1)}(Z_j-Z)^k,\qquad k=0,\dots,n \tag{3}$$

and

$$\sum_{j=1}^{M}\sigma_n^{(j-1)}(Z_j-Z)^k=0,\qquad k=0,\dots,n. \tag{4}$$

From (1), $P(Z)$ can be represented as follows:

$$P(Z)=P_1(Z)+P_2(Z) \tag{5a}$$

where

$$\begin{aligned}P_1(Z)&=\frac{1}{2\pi i}\int_\Gamma\frac{p(\zeta)\,d\zeta}{\zeta-Z}=\frac{1}{2\pi i}\sum_{j=1}^{M}P_j(Z)\int_{Z_j}^{Z_{j+1}}\frac{d\zeta}{\zeta-Z}\\&=-\frac{1}{2\pi i}\sum_{j=1}^{M}(P_j(Z)-P_{j-1}(Z))\ln|Z_j-Z|\\&\quad+\frac{1}{2\pi}\sum_{j=1}^{M}P_j(Z)\,\varphi_{j,j+1}(Z)\end{aligned} \tag{5b}$$

$$\begin{aligned}P_2(Z)&=-\frac{1}{2\pi i}\int_\Gamma\frac{p(\zeta)\,d\zeta}{\zeta-\hat Z}=-\frac{1}{2\pi i}\sum_{j=1}^{M}P_j(\hat Z)\int_{Z_j}^{Z_{j+1}}\frac{d\zeta}{\zeta-\hat Z}\\&=\frac{1}{2\pi i}\sum_{j=1}^{M}(P_j(\hat Z)-P_{j-1}(\hat Z))\ln|Z_j-\hat Z|\\&\quad-\frac{1}{2\pi}\sum_{j=1}^{M}P_j(\hat Z)\,\psi_{j,j+1}(\hat Z),\qquad \hat Z=(\bar Z)^{-1}\end{aligned} \tag{5c}$$

where $\varphi_{j,j+1}(Z)$ is the measure of angle from vector $\overrightarrow{ZZ}_j$ to $\overrightarrow{ZZ}_{j+1}$ and $\psi_{j,j+1}(\hat Z)$ is the measure of angle from vector $\overrightarrow{\hat ZZ}_j$ to $\overrightarrow{\hat ZZ}_{j+1}$, $\hat Z=\bar Z^{-1}$. We observe that

$$\begin{aligned}\cos\psi_{j,j+1}(Z)&=\frac{(x_j-x)(x_{j+1}-x)+(y_j-y)(y_{j+1}-y)}{|Z-Z_j|\,|Z-Z_{j+1}|}\\&=:f_1(Z,j)\end{aligned} \tag{5d}$$

$$\cos \psi_{j,j+1}(\hat{Z}) = \frac{(|Z|^2 x_j - x)(|Z|^2 x_{j+1} - x) + (|Z|^2 y_j - y)(|Z|^2 y_{j+1} - y)}{||Z|^2 Z_j - Z| \, ||Z|^2 Z_{j+1} - Z|}$$

$$=: f_2(Z, j) \tag{5e}$$

where $Z = x + iy$, $Z_j = x_j + iy_j$. Therefore, we have

$$\varphi_{j,j+1}(Z) = \cos^{-1} f_1(Z, j) \tag{5f}$$

$$\psi_{j,j+1}(\hat{Z}) = \cos^{-1} f_2(Z, j). \tag{5g}$$

We can also write $P_1(Z)$, $P_2(Z)$ in the following forms:

$$\begin{aligned} P_1(Z) &= \frac{1}{2\pi i} \sum_{j=1}^{M} p_j(Z) \ln \frac{Z_{j+1} - Z}{Z_j - Z}, \\ P_2(Z) &= \frac{-1}{2\pi i} \sum_{j=1}^{M} p_j(\hat{Z}) \ln \frac{Z_{j+1} - \hat{Z}}{Z_j - \hat{Z}}. \end{aligned} \tag{6}$$

Whichever branches of the logarithm are chosen, we should remember that the imaginary part of $\ln((Z_{j+1} - Z)/(Z_j - Z))$ is $\varphi_{j,j+1}(Z)$ and $\mathrm{Im}(\ln((Z_{j+1} - \hat{Z})/(Z_j - \hat{Z})))$ is $\psi_{j,j+1}(\hat{Z})$.

We know ([2, Theorem 3]) that any function belonging to $AC^{(n)}(\Gamma)$ can be approximated by complex spline functions.

Let $\mathscr{L}$ be the operator introduced in [2], $p = \mathscr{L}(F)$, where $F \in AC^{(n)}(\Gamma)$, then (see [2])

$$\|F^{(s)} - \mathscr{L}^{(s)}(F)\| \leqslant K_s \omega(F^{(n)}: |\Delta|) \, |\Delta|^{n-s}, \qquad 0 \leqslant s \leqslant n. \tag{7}$$

We shall prove the following:

Theorem 1. *Let F be analytic in U, $F \in A_B H^{(n)}(\bar{U})$, $n \geqslant 2$, and let $p = \mathscr{L}(F)$ be the pseudo-interpolation spline function. Then the complex harmonic spline*

$$P(Z) = \frac{1}{2\pi} \int_0^{2\pi} p(\zeta) \, \mathrm{Re}\left(\frac{\zeta + Z}{\zeta - Z}\right) d\theta \qquad \zeta = e^{i\theta}, \quad Z \in U$$

approximates F in U as follows:

$$|P(Z) - F(Z)| \leqslant \frac{K_0}{2\pi} \omega(F^{(n)}; |\Delta|) \, |\Delta|^n, \qquad Z \in \bar{U} \tag{8}$$

$$|P_Z(Z) - F'(Z)| \leqslant \left(\frac{\pi}{2} K_2 + K_1 |\Delta|\right) \omega(F^{(n)}; |\Delta|) \, |\Delta|^{n-2}, \qquad Z \in \bar{U} \tag{9}$$

$$|P_Z(Z)| = |P_Z(Z) - F_Z(Z)|$$
$$\leqslant \frac{\pi}{2}(K_2 + K_1 |\Delta|)\, \omega(F^{(n)}; |\Delta|)\, |\Delta|^{n-2}, \qquad Z \in \bar{U}. \quad (10)$$

$\omega(f; |\Delta|)$ is the modulus of continuity of f on Γ, where K_0, K_1, K_2 are constants given in (7).

This theorem tells us that if $|\Delta|$ tends to zero, the function P and its derivatives converge uniformly to F and its derivatives respectively on the closed disc $|Z| \leqslant 1$.

The proof of Theorem 1 is as follows.

Using the maximum modulus principle and (7) we have (8). Now

$$|P_Z(Z) - F'(Z)| \leqslant \operatorname*{Sup}_{Z_0 \in \Gamma} \left| \frac{1}{2\pi} \int_\Gamma \frac{\varphi(\zeta, Z_0)}{\zeta - Z}\, d\zeta + p'_k(Z_0) - F'(Z_0) \right|, \qquad Z \in \bar{U},$$

where

$$\varphi(\zeta, Z_0) = (p'(\zeta) - F'(\zeta)) - (p'(Z_0) - F'(Z_0))$$
$$|\varphi(\zeta, Z_0)| = \left| \int_{\widehat{Z_0\zeta}} (p^{(2)}(t) - F^{(2)}(t))\, dt \right|$$
$$\leqslant \max_{t \in \Gamma} |p^{(2)}(t) - F^{(2)}(t)|\, |\widehat{Z_0\zeta}|$$

where $|\widehat{Z_0\zeta}|$ is the arc length from Z_0 to ζ on Γ.

Hence we have

$$2 \int_{\tilde{\Gamma}_1} \left| \frac{\varphi(\zeta, Z_0)}{\zeta - Z_0} \right| |d\zeta| \geqslant \left| \int_\Gamma \frac{\varphi(\zeta, Z_0)}{\zeta - Z_0}\, d\zeta \right|$$

where $\tilde{\Gamma}_1$ is one of the half circles $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ such that

$$\left| \int_{\tilde{\Gamma}_1} \frac{\varphi(\zeta, Z_0)}{\zeta - Z_0}\, d\zeta \right| = \operatorname{Max} \left\{ \left| \int_{\tilde{\gamma}_1} \frac{\varphi(\zeta, Z_0)}{\zeta - Z_0}\, d\zeta \right|, \left| \int_{\tilde{\gamma}_2} \frac{\varphi(\zeta, Z_0)}{\zeta - Z_0}\, d\zeta \right| \right\}$$
$$\tilde{\gamma}_1 = \{ Z \mid Z = e^{i\theta},\ 0 \leqslant \theta \leqslant \pi \},$$
$$\tilde{\gamma}_2 = \{ Z \mid Z = e^{i\theta},\ \pi \leqslant \theta \leqslant 2\pi \}.$$

$\tilde{\Gamma}_1$ denotes $\tilde{\gamma}_1$ or $\tilde{\gamma}_2$, and in either case we stipulate that $|\widehat{Z_0\zeta}| \leqslant \pi$. Since $1 \leqslant |\widehat{Z_0\zeta}|/|Z_0 - \zeta| \leqslant \pi/2$, therefore,

$$2\int_{\Gamma_1} \frac{|\varphi(\zeta, Z_0)|}{|\zeta - Z_0|}\,|d\zeta| \leqslant \pi^2 \widehat{\underset{t\in\Gamma}{\mathrm{Max}}}\, |p^{(2)}(t) - F^{(2)}(t)|$$

$$|P_Z(Z) - F'(Z)| \leqslant \frac{\pi}{2} \underset{t\in\Gamma}{\mathrm{Max}}\, |p^{(2)}(t) - F^{(2)}(t)| + \underset{t\in\Gamma}{\mathrm{Max}}\, |p'(t) - F'(t)|$$

and from (7) we have (9).

Since

$$|P_{\bar Z}(Z) - F_{\bar Z}(Z)| \leqslant \frac{1}{2\pi} \underset{Z_0\in\Gamma}{\mathrm{Sup}} \left| \int_\Gamma \frac{\psi(\zeta, Z_0)\, d\zeta}{\zeta - Z_0} \right|$$

where

$$\psi(\zeta, Z_0) = (\zeta p'(\zeta) - F'(\zeta)) - (Z_0\, p'(Z_0) - Z_0 F'(Z_0))$$

$$|\psi(\zeta, Z_0)| \leqslant \underset{\zeta\in\Gamma}{\mathrm{Max}} \{ |p'(\zeta) - F'(\zeta)| + |p^{(2)}(\zeta) - F^{(2)}(\zeta)| \}\ |\widehat{Z_0\zeta}|$$

we have

$$|P_{\bar Z}(Z)| \leqslant \frac{\pi}{2} \underset{\zeta\in\Gamma}{\mathrm{Max}} \{ |p'(\zeta) - F'(\zeta)| + |p^{(2)}(\zeta) - F^{(2)}(\zeta)| \}$$

and from (7) we obtain (10). Q.E.D.

Remark 1. Results similar to those in Theorem 1 can be obtained when $\mathscr{L}(F)$ is replaced by $I(F)$, the interpolating complex spline (for the case of equidistantly spaced node points, see [1]).

COROLLARY. *Let $F(Z)$ and $P(Z)$ be defined as in Theorem* 1. *Then*

$$\left| \frac{\partial^j P(Z)}{\partial Z^j} - F^{(j)}(Z) \right| \leqslant \omega\, |\Delta|^{n-j-1} \left[\frac{\pi}{2} K_{j+1} + K_j\, |\Delta| \right], \qquad Z \in \bar U,\ j = \overline{1, n-1}$$

where $\omega = \omega(F^{(n)}; |\Delta|)$, K_j, K_{j+1} are constants given in (7).

The following theorem gives conditions under which the complex harmonic spline is an open mapping.

THEOREM 2. *Let F be analytic in U, $F \in A_B H^{(n)}(\bar U)$, $n \geqslant 2$. Assume that $F^{(1)}(Z) \neq 0$, $Z \in \bar U$, choose $|\Delta|$ sufficiently small that*

$$\begin{aligned} &\omega < \underset{Z\in\Gamma}{\mathrm{Min}}\, |F'(Z)|/\pi(K_2 + 2K_1), && n = 2 \\ &\omega^{1/(n-1)}\, |\Delta| < [\underset{Z\in\Gamma}{\mathrm{Min}}\, |F'(Z)|/\pi(K_2 + 2K_1)]^{1/(n-2)}, && n > 2 \end{aligned} \qquad (11)$$

where $\omega = \omega(F^{(n)}; |\Delta|)$ *is the modulus of continuity of* $F^{(n)}$ *on* Γ. *Set*

$$P(Z) = \frac{1}{2\pi} \int_0^{2\pi} p(\zeta) \operatorname{Re}\left(\frac{\zeta+Z}{\zeta-Z}\right) d\theta, \qquad \zeta = e^{i\theta}, Z \in \bar{U}$$

with $p = \mathscr{L}(F)$. *Then the Jacobian* J *of* P *is positive on* $\bar{U}$.

Proof. Let

$$\eta = \left(\frac{\pi}{2} K_2 + K_1 |\Delta|\right) \omega |\Delta|^{n-2}, \qquad \xi = \frac{\pi}{2}(K_2 + K_1 |\Delta|) \omega |\Delta|^{n-2}. \tag{12}$$

Since $|\Delta| < 2$, from (9), (10) we have

$$J(Z) = |P_Z|^2 - |P_{\bar{Z}}|^2 \geqslant (|F'(Z)| - \eta)^2 - \xi^2.$$

From (11), we obtain $J(Z) > 0$. Q.E.D.

It is easy to prove the following

LEMMA. *Let* γ *be a closed Jordan curve*; γ *is a homeomorphic image of* Γ, $\gamma = f(\Gamma)$, $f \in C^{(1)}(\Gamma)$. *If* $f'(Z) \neq 0$ *for* $Z \in \Gamma$, *then*

$$\begin{aligned} m_f &:= \inf_{Z_1, Z_2 \in \Gamma} \left|\frac{f(Z_1) - f(Z_2)}{Z_1 - Z_2}\right| > 0, \\ M_f &:= \operatorname*{Sup}_{Z_1, Z_2 \in \Gamma} \left|\frac{f(Z_1) - f(Z_2)}{Z_1 - Z_2}\right| < \infty. \end{aligned} \tag{13}$$

THEOREM 3. *Let* D *be a simply connected domain,* $\partial D = \gamma$ *a closed Jordan curve with bounded curvature,* $W = F(Z)$ *a conformal mapping of* U *onto* D, *and* $F \in AH^{(n)}(\bar{U})$, $n \geqslant 2$. *If* $|\Delta|$ *is so small that*

$$\begin{aligned} &\omega < m_f/\pi(K_2 + 2K_1), \qquad n = 2; \\ &\omega^{1/(n-2)} |\Delta| < (m_F/\pi(K_2 + 2K_1))^{1/(n-2)}, \qquad n > 2 \end{aligned} \tag{14}$$

where $\omega = \omega(F^{(n)}; |\Delta|)$, m_F *defined as in* (13), *then the function*

$$P(Z) = \frac{1}{2\pi} \int_0^{2\pi} \mathscr{L}(F) \operatorname{Re}\left(\frac{\zeta+Z}{\zeta-Z}\right) d\theta$$

maps U *onto a simply connected domain* D_p. *This mapping is* 1–1 *and sense-preserving*; *moreover*,

$$\lim_{|\Delta| \to 0} D_p = D \tag{15}$$

in Caratheodory's sense.

Remark 2. Theorems 2 and 3 remain valid if $\mathscr{L}(F)$ is replaced by $I(F)$ (see Remark 1).

Proof of Theorem 3. Since $F \in C^{(n)}(\bar{U})$, $\gamma = F(\Gamma)$ is smooth, $\arg F'(\zeta)$ is continuous in $\bar{U}$ and $\arg F'(\zeta) = \varphi(\zeta) - \arg \zeta - \pi/2$, where $\varphi(\zeta) = \arg(dF(e^{i\theta})/d\theta)$. Let S denote the arc length of γ. Then

$$|S_2 - S_1| = \int_{\theta_1}^{\theta_2} |F'(e^{i\theta})|\, d\theta \leqslant \left(\int_{\theta_1}^{\theta_2} |F'(e^{i\theta})|^2\, d\theta\right)^{1/2} \left(\int_{\theta_1}^{\theta_2} d\theta\right)^{1/2}$$
$$\leqslant K_1(\theta_2 - \theta_1)^{1/2}.$$

Let $K(\zeta)$ be the curvature of γ at points ζ. From the hypothesis, $K(\zeta) \leqslant K_0 < \infty$, where K_0 is a constant, hence

$$|\arg F'(\zeta_2) - \arg F'(\zeta_1)|$$
$$= \left| \int_{S_1}^{S_2} \frac{d\varphi}{dS}\, dS + (\theta_1 - \theta_2) \right| \leqslant K_0\, |S_2 - S_1| + |\theta_2 - \theta_1|$$
$$\leqslant K_0 K_1\, |\theta_2 - \theta_1|^{1/2} + |\theta_2 - \theta_1| \leqslant K\, |\theta_2 - \theta_1|^{1/2}.$$

Since $F'(Z) \neq 0$ for $Z \in U$, $\ln F'$ is analytic in U, and there exists a constant K' such that

$$|\ln F'(Z_2) - \ln F'(Z_1)| \leqslant K'\, |Z_1 - Z_2|^{1/2}, \qquad Z_1, Z_2 \in \bar{U}$$

[4, Chap. 9, Sect. 5, Theorem 4, 5]; therefore $\ln F'$ is continuous and hence bounded in $\bar{U}$. We then conclude that

$$F'(Z) \neq 0, \qquad Z \in \Gamma;$$

hence from the lemma we have $m_F > 0$.

The directional derivative of a complex harmonic function G can be written as $\partial G/\partial l_\theta = G_Z e^{i\theta} + G_{\bar{Z}} e^{-i\theta}$, and this is a complex harmonic function. From Theorem 1 we obtain

$$\sup_{Z \in \bar{U}} \left| \frac{\partial P}{\partial \theta} - \frac{\partial F}{\partial \theta} \right| \leqslant \sup_{Z \in \Gamma} |P_Z - F'| + \sup_{Z \in \Gamma} |P_{\bar{Z}}| \leqslant \eta + \xi$$

where η and ξ are defined by (12).

From (9), (10) we obtain

$$\left| \frac{P(Z_2) - P(Z_1)}{Z_2 - Z_1} - \frac{F(Z_2) - F(Z_1)}{Z_2 - Z_1} \right| \leqslant \eta + \xi;$$

therefore,

$$|Z_2 - Z_1|(m_F - \eta - \xi) \leqslant |P(Z_2) - P(Z_1)| \leqslant (M_F + \eta + \xi)\,|Z_2 - Z_1| \quad (16)$$

from (14). $\eta + \xi < m_F$. We conclude that

$$P(Z_2) = P(Z_1) \qquad \text{if and only if} \quad Z_1 = Z_2;$$

i.e., $W = P(Z)$ is a homeomorphic mapping. Note that

$$m_F \leqslant \min_{Z \in \Gamma} |F'(Z)|. \tag{17}$$

From (17) and (14) we obtain inequalities (11). Theorem 2 tells us

$$J(Z) > 0 \qquad \text{for} \quad Z \in \bar{U}. \tag{18}$$

The mapping $W = P(Z)$ from U onto D_P is sense-preserving [4].

Since the Jacobian is positive for $Z \in \bar{U}$, the mapping $W = P(Z)$ is open; thus $D_P = P(U)$ is a domain, and no interior point of U can be mapped onto the boundary of D_P. Hence the boundary of D_P must be the image of Γ. Since (16) is valid on $\bar{U}$, P is a homeomorphic mapping from Γ to $\gamma = \partial D_P$. We thus conclude that γ is a closed Jordan curve and D_P is a simply connected domain.

If $Z \in \Gamma$, from Theorem 1 we have

$$|P(Z) - F(Z)| < \frac{K_0}{2\pi}\,\omega(F^{(n)}; |\Delta|)\,|\Delta|^n;$$

therefore (15) is proved. Q.E.D.

Since $P_Z, P_{\bar{Z}}$ are continuous in $\bar{U}$, denote the complex dilatation by $\chi(Z) := P_{\bar{Z}}(Z)/P_Z(Z)$, $Z \in \bar{U}$, and

$$D(Z) = [|P_Z(Z)| + |P_{\bar{Z}}(Z)|]/[|P_Z(Z)| - |P_{\bar{Z}}(Z)|].$$

We can prove the following

THEOREM 4. *Let F be a conformal mapping of U onto D, $F \in AH^{(n)}(\bar{U})$. Let P be a complex harmonic spline defined as in Theorem* 1, *and choose $|\Delta|$ so small that*

$$\lambda = \pi(K_2 + 2K_1)\,\omega(F^{(n)}; |\Delta|)\,|\Delta|^{n-2} < m_F, \qquad n \geqslant 2. \tag{18}$$

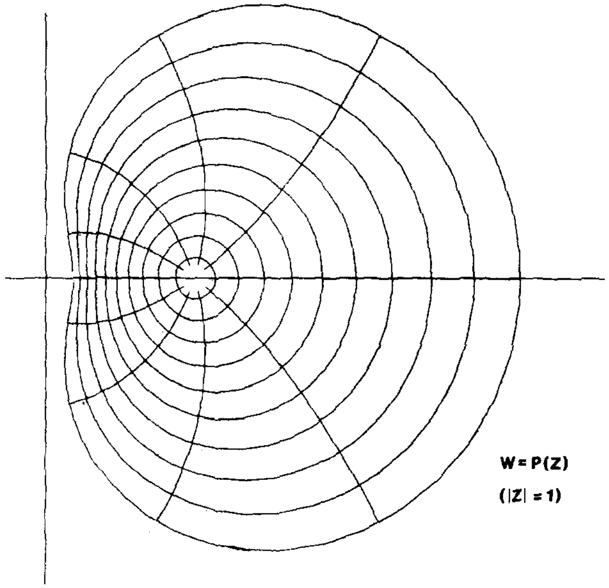

FIG. 1. This is the image of a family of curves which consists of concentric circles $|Z| = r = (j-1)/10$ $(j = \overline{2, 10})$ and radii $\theta = 2(j-1)\pi/10$ $(j = \overline{1, 10})$ under the mapping $W = (1/2\pi) \int_0^{2\pi} p(\zeta) \operatorname{Re}[(\zeta + Z)/(\zeta - Z)]\, d\theta$.

Let ε be any positive number satisfying the relation

$$\lambda < \varepsilon \leqslant m_F. \tag{19}$$

Then P satisfies the Beltrami differential equation

$$W_{\bar{Z}} = \chi W_Z \tag{20}$$

and χ is continuous in $\bar{U}$. Further, P is a K-quasiconformal mapping with dilatation

$$D(Z) < K = \frac{M+\varepsilon}{M-\varepsilon}, \qquad |\chi(Z)| < \frac{\varepsilon}{M} \tag{21}$$

where

$$M := \operatorname*{Sup}_{Z \in \Gamma} |F'(Z)|.$$

Proof. From (8), (9) we have

$$|D(Z)| \leqslant \frac{|F'(Z)| + \eta + \xi}{|F'(Z)| - \eta - \xi},$$

with η, ξ as defined by (12). From (18), (19)

$$|D(Z)| < \frac{M+\lambda}{M-\lambda} < \frac{M+\varepsilon}{M-\varepsilon} = K$$

and

$$|\chi(Z)| < \frac{K-1}{K+1} = \frac{\varepsilon}{M}.$$

Then (21) is proved.

Now $\lambda < m_F$, (14) is valid, from Theorem 3, $W = P(Z)$ is a homeomorphism and the function P satisfies (20); obviously, $\chi(Z)$ is continuous in $\bar{U}$. Q.E.D.

Figure 1 is a set of curves which are the image of radii and concentric circles under the mapping

$$W = P(Z) = \frac{1}{2\pi} \int_0^{2\pi} p(\zeta) \operatorname{Re}\left(\frac{\zeta + Z}{\zeta - Z}\right) d\theta, \qquad \zeta = e^{i\theta},\ |Z| \leqslant 1,$$

where p is the interpolating complex cubic spline function for $F(Z) = (Z-3)^4 + 8/(Z-5) + 12/(Z-5)^2$, the knots of the cubic spline p are $Z_j = e^{i(2\pi j/20)}$, $j = 1,..., 20$, and $p(Z_j) = F(Z_j)$, $j = 1,..., 20$.

The boundary curve is $W = P(Z)$, $|Z| = 1$. We see that the two families of curves are almost perpendicular. The reason is that since $W = F(Z)$ is a conformal mapping on $\bar{U}$, then $W = P(Z)$ is almost a conformal mapping on $\bar{U}$. The theoretical demonstration of this fact is given in the proofs of Theorems 1–4.

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